

Quantum interference effect on the density of states in disordered d -wave superconductors

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Abstract

The quantum interference effect on the quasiparticle density of states (DOS) is studied with the diagrammatic technique in two-dimensional d -wave superconductors with dilute nonmagnetic impurities both near the Born and near the unitary limits. We derive in details the expressions of the Goldstone modes (cooperon and diffuson) for quasiparticle diffusion. The DOS for generic Fermi surfaces is shown to be subject to a quantum interference correction of logarithmic suppression, but with various renormalization factors for the Born and unitary limits. Upon approaching the combined limit of unitarity and nested Fermi surface, the DOS correction is found to become a δ -function of the energy, which can be used to account for the resonant peak found by the numerical studies.

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I. Introduction

Since the experiments revealed the d -wave symmetry of the order parameter in cuprate superconductors [1], the physics of low-energy quasiparticle excitations in disordered two-dimensional (2D) d -wave superconductors has been a subject of ongoing

intensive research [2]. The characteristic feature of the $d_{x^2-y^2}$ -wave pairing state is the existence of four nodal points where the order parameter vanishes. In the vicinity of the gap nodes there exist low-lying Dirac-type quasiparticle excitations. An understanding of the disorder effect induced by randomly-distributed impurities on these low-energy states is essential for the elucidation of the thermodynamic and transport properties of disordered d -wave superconductors. During the years a number of theoretical approaches, such as the self-consistent approximation schemes [3–9], non-perturbative methods [10–17], and numerical studies [18–22], have been developed to calculate the quasiparticle density of states (DOS) in the presence of disorder. Unfortunately, many of these theories contradict each other. The DOS at zero energy was variously predicted to be finite [3–10], vanishing [11–14,18,21], and divergent [15,20]. Recently, it has been made clear that [16,17,19,22] much of this controversy roots in the fact that the d -wave superconductor is fundamentally sensitive to the details of disorder, as well as to certain symmetries of the normal state Hamiltonian.

In order to clarify the physics of the various asymptotic results for the DOS, Yashenkin et al. [23] analyzed the diffusive modes in disordered d -wave superconductors and calculated the weak-localization correction to the DOS with the diagrammatic technique. It is well known that the weak localization in electron systems is a manifestation of the quantum interference (QI) effect, which stems from the existence of the Goldstone modes (cooperon and diffuson) [24]. As pointed out by Altland and Zirnbauer [25], the Goldstone modes in a superconductor have different features from those in a normal metal, for the local particle-hole symmetry of the superconducting state gives rise to a combination of impurity- and Andreev-scattering processes. As a result, every cooperon or diffuson mode in the retarded-advanced (RA) channel entails a corresponding mode in the retarded-retarded (RR) or advanced-advanced (AA) channel. In the combined limit of the unitarity and nested Fermi surface (the UN limit), each of these 0-mode cooperon and diffuson has a π -mode counterpart due to the global particle-hole symmetry [23,26]. For disordered d -wave superconductors, it was found [23] that in generic situations the quasiparticle DOS is subject to a loga-

rithmic weak-localization correction due to the 0-mode cooperon, and the existence of diffusive π -modes can give rise to a finite, or even divergent zero-energy DOS upon approaching the UN limit.

The weak-localization calculations carried out by Yashenkin et al. appear to capture the physical origin of the discrepancies between predicted low-energy quasiparticle DOS. Furthermore, the quasiparticle weak-localization effect was also suggested to have important influences on transport properties such as the electrical [6], spin [13], thermal [27], and optical [28] conductivities of d -wave superconductors. While the QI effects have been widely investigated for disordered normal metals [24], a corresponding theory for random Dirac fermions in superconducting cuprates is far from well established, and thus highly deserves further development.

This paper presents an intensive study of the QI effect on the quasiparticle DOS in 2D d -wave superconductors with dilute nonmagnetic impurities both near the Born and near the unitary limits. These two limiting cases are considered to be closely related with the disorder effects in cuprate superconductors. It is reasonable that the disorder due to defects off the copper-oxygen plane may be treated in the Born approximation and that defects in the plane may be in the unitary limit [6]. Albeit sharing certain aspects with Ref. [23], we further develop the weak-localization theory in d -wave superconductors, and obtain some new results in this paper. First, the expressions of the Goldstone modes for quasiparticle diffusion are derived in details both near the Born and near the unitary limits. Second, we calculate the additional contributions to the DOS from those lowest-order self-energy diagrams containing non-singular ladders, which were not taking into account previously. These diagrams are shown to give rise to various renormalization factors for the DOS correction in the Born and unitary limits. Third, by taking into account a new nontrivial self-energy diagram with the π -mode diffuson, we show that the QI correction to the DOS becomes a δ -function of the energy upon approaching the UN limit. This result can be used to account for the resonant peak found by the previous numerical studies [19, 20].

The structure of this paper is as follows. In Sec. II, the commonly used self-

consistent T -matrix approximation (SCTMA) is described for a weakly-disordered $d_{x^2-y^2}$ -wave superconductor. Using the SCTMA, we derive in Sec. III the expressions of the 0-mode and π -mode cooperons and diffusons. The QI correction to the quasiparticle DOS is calculated in Sec. IV, and the conclusions are summarized in Sec. V. The Appendix provides some mathematical formulas, which are useful for our derivations.

II. SCTMA for d-wave superconductors

Let us consider a most extensively studied model for a 2D $d_{x^2-y^2}$ -wave superconductor. In the tight-binding approximation, the normal-state dispersion of a square lattice is given by $\xi_{\mathbf{k}} = -t(\cos k_x a + \cos k_y a) - \mu$ where a is the lattice constant, t is the nearest-neighbor hopping integral, and μ is the chemical potential. The nested Fermi surface corresponds to the half-filling case ($\mu = 0$). The order parameter of the $d_{x^2-y^2}$ -wave pairing state can be expressed by $\Delta_{\mathbf{k}} = \Delta_0(\cos k_x a - \cos k_y a)$. The four gap nodes are given by $\mathbf{k}_n = \pm(k_0, \pm k_0)$ with $k_0 = (1/a) \arccos(-\mu/2t)$, which satisfy $\xi_{\mathbf{k}_n} = \Delta_{\mathbf{k}_n} = 0$. In the vicinity of these nodes, the quasiparticle spectrum can be linearized as $\epsilon_{\mathbf{k}} = \sqrt{\xi_{\mathbf{k}}^2 + \Delta_{\mathbf{k}}^2} \approx \sqrt{(\mathbf{v}_f \cdot \tilde{\mathbf{k}})^2 + (\mathbf{v}_g \cdot \tilde{\mathbf{k}})^2}$, where $\mathbf{v}_f = (\partial \xi_{\mathbf{k}} / \partial \mathbf{k})_{\mathbf{k}_n}$, $\mathbf{v}_g = (\partial \Delta_{\mathbf{k}} / \partial \mathbf{k})_{\mathbf{k}_n}$, and $\tilde{\mathbf{k}}$ is the momentum measured from the node \mathbf{k}_n . A direct calculation yields $v_f = v_g t / \Delta_0 = \sqrt{2} t a \sqrt{1 - (\mu/2t)^2}$.

In the presence of randomly-distributed nonmagnetic impurities, the time-reversal and spin-rotational symmetries remain preserved. Thus the system belongs to symmetry class CI in the classification of Ref. [25]. The impurity potential is assumed to be point-like, meaning that the intra- and inter-node scatterings are described by a single potential strength V . For a low impurity concentration n_i , the quasiparticle self-energy in the SCTMA [23] can be expressed in the Nambu spinor representation as

$$\Sigma^{R(A)}(\epsilon) = n_i T^{R(A)}(\epsilon) = (\lambda \epsilon \mp i \gamma) \tau_0 + \eta \gamma \tau_3, \quad (2.1)$$

for $|\epsilon| \ll \gamma$. Here λ is the mass renormalization factor, γ is the impurity-induced relaxation rate, η is a dimensionless parameter, $\eta \gamma$ represents the decrease of the chemical potential induced by the impurity scatterings, and τ_0 and τ_i ($i = 1, 2, 3$) stand for the

2×2 unity and Pauli matrices, respectively. A use of Dyson's equation immediately yields the impurity-averaged one-particle Green's functions as

$$G_{\mathbf{k}}^{R(A)}(\epsilon) = \frac{[(1-\lambda)\epsilon \pm i\gamma]\tau_0 + \Delta_{\mathbf{k}}\tau_1 + \xi_{\mathbf{k}}\tau_3}{[(1-\lambda)\epsilon \pm i\gamma]^2 - \epsilon_{\mathbf{k}}^2}, \quad (2.2)$$

where the shift of chemical potential has been absorbed in μ . The zero-energy density of states is calculated as $\rho_0 = -(1/\pi)\text{Im} \sum_{\mathbf{k}} \text{Tr} G_{\mathbf{k}}^R = 4l\gamma/\pi^2 v_f v_g$, where $G_{\mathbf{k}}^{R(A)} = G_{\mathbf{k}}^{R(A)}(0)$ and $l = \ln(\Gamma/\gamma) > 1$ with $\Gamma \sim \sqrt{v_f v_g}/a$.

The parameters γ , λ , and η can be evaluated consistently via the T -matrix equation

$$T^{R(A)}(\epsilon) = V\tau_3 + V\tau_3 g^{R(A)}(\epsilon) T^{R(A)}(\epsilon), \quad (2.3)$$

with $g^{R(A)}(\epsilon) = \sum_{\mathbf{k}} G_{\mathbf{k}}^{R(A)}(\epsilon)$. Using Eq. (2.2), we can show that

$$g^{R(A)}(\epsilon) = \frac{\pi\rho_0}{2\gamma} [(\lambda-1)(1-l^{-1})\epsilon \mp i\gamma] \tau_0 + (V^{-1} - U^{-1}) \tau_3, \quad (2.4)$$

for $|\epsilon| \ll \gamma$, where U is the effective impurity potential given by $U^{-1} = V^{-1} + \sum_{\mathbf{k}} \xi_{\mathbf{k}}/(\epsilon_{\mathbf{k}}^2 + \gamma^2)$. A substitution of Eqs. (2.1) and (2.4) into Eq. (2.3) leads to $\gamma = 2n_i/\pi\rho_0(1+\eta^2)$, $\lambda = (1-\eta^2)(l-1)/(\eta^2+2l-1)$, and $\eta = 2/\pi\rho_0 U$.

The Born limit corresponds to $\eta^2 \gg 2l$, yielding

$$\gamma = \frac{\pi}{2} n_i \rho_0 U^2, \quad \lambda = 1-l; \quad (2.5)$$

and the unitary limit corresponds to $\eta \rightarrow 0$, meaning that

$$\gamma = \frac{2n_i}{\pi\rho_0}, \quad \lambda = \frac{l-1}{2l-1}. \quad (2.6)$$

Throughout this paper, we mainly consider the cases near either the Born or the unitary limit. It is worthy to point out that the values of the impurity potential V driving the system into these two limits are dependent on the band structure.

III. The diffusive modes

Since the QI effect is related to the diffusive modes, we first derive the expressions of the 0-mode and π -mode cooperons and diffusons for a disordered d -wave superconductor.

A. 0-mode cooperon and diffuson

The 0-mode cooperon and diffuson exist both in RA and in RR channels due to the local particle-hole symmetry $\tau_2 G_{\mathbf{k}}^R(\epsilon) \tau_2 = -G_{\mathbf{k}}^A(-\epsilon)$. The ladder diagrams for the cooperon are given by Fig. 1(b) in Ref. [23]. The equation for 0-mode cooperon can be expressed as

$$\mathcal{C}(\mathbf{q}; \epsilon, \epsilon') = \mathcal{W}(\epsilon, \epsilon') + \mathcal{W}(\epsilon, \epsilon') \mathcal{H}(\mathbf{q}; \epsilon, \epsilon') \mathcal{C}(\mathbf{q}; \epsilon, \epsilon'), \quad (3.1)$$

where the two-particle irreducible vertex $\mathcal{W}(\epsilon, \epsilon')$ and the integral kernel $\mathcal{H}(\mathbf{q}; \epsilon, \epsilon')$ are defined in the RR and RA channels as

$$\mathcal{W}(\epsilon, \epsilon')^{RR(A)} = n_i T^R(\epsilon) \otimes T^{R(A)}(\epsilon') \quad (3.2)$$

and

$$\mathcal{H}(\mathbf{q}; \epsilon, \epsilon')^{RR(A)} = \sum_{\mathbf{k}} G_{\mathbf{q}-\mathbf{k}}^R(\epsilon) \otimes G_{\mathbf{k}}^{R(A)}(\epsilon'). \quad (3.3)$$

Equation (3.1) can be also expressed as

$$\mathcal{A}(\mathbf{q}; \epsilon, \epsilon') \mathcal{C}(\mathbf{q}; \epsilon, \epsilon') = \mathcal{W}(\epsilon, \epsilon'), \quad (3.4)$$

where

$$\mathcal{A}(\mathbf{q}; \epsilon, \epsilon') = \mathcal{I} - \mathcal{W}(\epsilon, \epsilon') \mathcal{H}(\mathbf{q}; \epsilon, \epsilon'), \quad (3.5)$$

with $\mathcal{I} = \tau_0 \otimes \tau_0$. From Eqs. (3.1)–(3.5), it follows that one can make a decomposition of $\mathcal{X} = \sum_{ij} X_{ij} \tau_i \otimes \tau_j$ for $\mathcal{X} = \mathcal{W}, \mathcal{H}, \mathcal{A}$, and \mathcal{C} . Substituting Eq. (2.1) into Eq. (3.2), we can easily obtain all the nonvanishing components of $\mathcal{W}(\epsilon, \epsilon')$ as

$$W(\epsilon, \epsilon')_{00}^{RR(A)} = \mp \frac{2\gamma}{\pi \rho_0 (1 + \eta^2)} \left[1 + i \frac{\lambda}{\gamma} (\epsilon \pm \epsilon') \right], \quad (3.6)$$

$$W(\epsilon, \epsilon')_{33}^{RR(A)} = \frac{2\gamma \eta^2}{\pi \rho_0 (1 + \eta^2)}, \quad (3.7)$$

$$W(\epsilon, \epsilon')_{03}^{RR(A)} = -i \frac{2\gamma \eta}{\pi \rho_0 (1 + \eta^2)} \left(1 + i \frac{\lambda}{\gamma} \epsilon \right), \quad (3.8)$$

and

$$W(\epsilon, \epsilon')_{30}^{RR(A)} = \mp i \frac{2\gamma\eta}{\pi\rho_0(1+\eta^2)} \left(1 \pm i \frac{\lambda}{\gamma} \epsilon'\right). \quad (3.9)$$

It then follows that the dominant component of $\mathcal{W}(\epsilon, \epsilon')$ near the Born limit ($\eta^2 \gg 2l$) is $W(\epsilon, \epsilon')_{33}$, while that near the unitary limit ($\eta^2 \ll 1$) is $W(\epsilon, \epsilon')_{00}$.

Now let us calculate $\mathcal{H}(\mathbf{q}; \epsilon, \epsilon')$ and $\mathcal{A}(\mathbf{q}; \epsilon, \epsilon')$. For small values of \mathbf{q} , ϵ , and ϵ' , we have

$$G_{\mathbf{q}-\mathbf{k}}^R(\epsilon) \approx G_{\mathbf{k}}^R + \epsilon \frac{\partial}{\partial \epsilon'} G_{\mathbf{k}}^R(\epsilon') \big|_{\epsilon'=0} - \mathbf{q} \cdot \nabla G_{\mathbf{k}}^R + \frac{1}{2} \mathbf{q} \mathbf{q} : \nabla \nabla G_{\mathbf{k}}^R \quad (3.10)$$

and

$$G_{\mathbf{k}}^{R(A)}(\epsilon') \approx G_{\mathbf{k}}^{R(A)} + \epsilon' \frac{\partial}{\partial \epsilon} G_{\mathbf{k}}^{R(A)}(\epsilon) \big|_{\epsilon=0}. \quad (3.11)$$

Substituting Eqs. (3.10) and (3.11) into Eq. (3.3), and using Eqs. (A1)–(A4) in the Appendix, we can show that all the nonvanishing components of $\mathcal{H}(\mathbf{q}; \epsilon, \epsilon')$ are given by

$$H(\mathbf{q}; \epsilon, \epsilon')_{00}^{RR(A)} = \mp \frac{\pi\rho_0}{4l\gamma} \left(1 - \frac{v_f^2 + v_g^2}{12\gamma^2} q^2\right), \quad (3.12)$$

$$H(\mathbf{q}; \epsilon, \epsilon')_{11}^{RR(A)} = \frac{\pi\rho_0}{8l\gamma} \left[(2l-1) + i \frac{1-\lambda}{\gamma} (\epsilon \pm \epsilon') - \frac{v_f^2 + 3v_g^2}{12\gamma^2} q^2\right], \quad (3.13)$$

and

$$H(\mathbf{q}; \epsilon, \epsilon')_{33}^{RR(A)} = \frac{\pi\rho_0}{8l\gamma} \left[(2l-1) + i \frac{1-\lambda}{\gamma} (\epsilon \pm \epsilon') - \frac{v_g^2 + 3v_f^2}{12\gamma^2} q^2\right]. \quad (3.14)$$

The diagonal components of $\mathcal{A}(\mathbf{q}; \epsilon, \epsilon')$ can be easily calculated by a substitution of Eqs. (3.6), (3.7), and (3.12)–(3.14) into Eq. (3.5), yielding

$$\begin{aligned} A(\mathbf{q}; \epsilon, \epsilon')_{00}^{RR(A)} = & -\frac{1}{4l(1+\eta^2)} \left[-2(2l-1) - (2l+1)\eta^2 \right. \\ & \left. + i \frac{2\lambda + (1-\lambda)\eta^2}{\gamma} (\epsilon \pm \epsilon') - \frac{2v_f^2 + 2v_g^2 + (3v_f^2 + v_g^2)\eta^2}{12\gamma^2} q^2 \right], \end{aligned} \quad (3.15)$$

$$A(\mathbf{q}; \epsilon, \epsilon')_{11}^{RR(A)} = \pm \frac{1}{4l(1+\eta^2)} \left[(2l-1) + i \frac{2\lambda(l-1) + 1}{\gamma} (\epsilon \pm \epsilon') - \frac{v_f^2 + 3v_g^2}{12\gamma^2} q^2 \right], \quad (3.16)$$

$$A(\mathbf{q}; \epsilon, \epsilon')_{22}^{RR(A)} = \frac{\eta^2}{4l(1+\eta^2)} \left[(2l-1) + i \frac{1-\lambda}{\gamma} (\epsilon \pm \epsilon') - \frac{v_f^2 + 3v_g^2}{12\gamma^2} q^2 \right], \quad (3.17)$$

and

$$A(\mathbf{q}; \epsilon, \epsilon')_{33}^{RR(A)} = \pm \frac{1}{4l(1 + \eta^2)} \left[(2l - 1) + 2\eta^2 + i \frac{2\lambda(l - 1) + 1}{\gamma} (\epsilon \pm \epsilon') - \frac{v_g^2 + 3v_f^2 + (2v_f^2 + 2v_g^2)\eta^2}{12\gamma^2} q^2 \right]. \quad (3.18)$$

We note that all the non-diagonal components of $\mathcal{H}(\mathbf{q}; \epsilon, \epsilon')$ are vanishing, and those of $\mathcal{W}(\epsilon, \epsilon')$ are negligible near the Born or unitary limit. Then Eq. (3.1) or (3.4) indicates that only the diagonal components of $\mathcal{C}(\mathbf{q}; \epsilon, \epsilon')$ may be singular for these two limits. As a result, the 0-mode cooperon can be expressed as $\mathcal{C}(\mathbf{q}; \epsilon, \epsilon') = \sum_i C(\mathbf{q}; \epsilon, \epsilon')_{ii} \tau_i \otimes \tau_i$. As will be shown below, all $C(\mathbf{q}; \epsilon, \epsilon')_{ii}$ are of diffusive poles. Obviously, only the diagonal components of $\mathcal{A}(\mathbf{q}; \epsilon, \epsilon')$ and $\mathcal{W}(\epsilon, \epsilon')$ are needed for the calculation of 0-mode cooperon, and thus Eq. (3.4) becomes equivalent to the following group of equations:

$$A_{00}C_{00} + A_{11}C_{11} + A_{22}C_{22} + A_{33}C_{33} = W_{00}, \quad (3.19)$$

$$A_{00}C_{11} + A_{11}C_{00} - A_{22}C_{33} - A_{33}C_{22} = 0, \quad (3.20)$$

$$A_{00}C_{22} - A_{11}C_{33} + A_{22}C_{00} - A_{33}C_{11} = 0, \quad (3.21)$$

$$A_{00}C_{33} - A_{11}C_{22} - A_{22}C_{11} + A_{33}C_{00} = W_{33}, \quad (3.22)$$

where the arguments $(\mathbf{q}, \epsilon, \text{ and } \epsilon')$ of A_{ii} , C_{ii} , and W_{ii} have been omitted.

Near the Born limit ($\eta^2 \gg 2l$), the singular terms in Eqs. (3.19)–(3.22) satisfy the following relations:

$$(2l + 1)C_{00}^{RR(A)} + (2l - 1)C_{22}^{RR(A)} \pm 2C_{33}^{RR(A)} = 0, \quad (3.23)$$

$$(2l + 1)C_{11}^{RR(A)} - (2l - 1)C_{33}^{RR(A)} \mp 2C_{22}^{RR(A)} = 0, \quad (3.24)$$

$$(2l + 1)C_{22}^{RR(A)} + (2l - 1)C_{00}^{RR(A)} \mp 2C_{11}^{RR(A)} = 0, \quad (3.25)$$

$$(2l+1)C_{33}^{RR(A)} - (2l-1)C_{11}^{RR(A)} \pm 2C_{00}^{RR(A)} = 0, \quad (3.26)$$

the only solution of which is given by

$$C_{00}^{RR(A)} = \mp C_{11}^{RR(A)} = -C_{22}^{RR(A)} = \mp C_{33}^{RR(A)}. \quad (3.27)$$

Substituting Eqs. (3.7), (3.15)–(3.18), and (3.27) into Eq. (3.22), and using Eq. (2.5), we can show that the terms of order η^{-2} cancel exactly out, leading to

$$C(\mathbf{q}; \epsilon, \epsilon')_{00}^{RR(A)} = \mp \frac{4\gamma^2}{\pi\rho_0} \frac{1}{Dq^2 - i(\epsilon \pm \epsilon')}, \quad (3.28)$$

where $D = (v_f^2 + v_g^2)/4l\gamma$ is the quasiparticle diffusion coefficient. Combining Eq. (3.27) with Eq. (3.28), and noting that the 0-mode diffuson has the same expression as that of the 0-mode cooperon due to the time-reversal symmetry, we obtain

$$\mathcal{C}(\mathbf{q}; \epsilon, \epsilon')^{RR(A)} = \mathcal{D}(\mathbf{q}; \epsilon, \epsilon')^{RR(A)} = \frac{4\gamma^2}{\pi\rho_0} \frac{1}{Dq^2 - i(\epsilon \pm \epsilon')} \left(\mp \tau_0 \otimes \tau_0 + \tau_1 \otimes \tau_1 \pm \tau_2 \otimes \tau_2 + \tau_3 \otimes \tau_3 \right), \quad (3.29)$$

which is in agreement with that appearing in Ref. [26] (aside from a disputed pre-factor). Similarly, we can show that Eq. (3.29) is also valid near the unitary limit ($\eta^2 \ll 1$). The above evaluations indicate that any small deviations from either limit do not make the Goldstone 0-modes gapped. However, in the intermediate region far from these two limits, the non-diagonal components of $\mathcal{W}(\epsilon, \epsilon')$ cannot be neglected, and thus $\mathcal{C}(\mathbf{q}; \epsilon, \epsilon')$ may contain some singular non-diagonal components.

B. π -mode cooperon and diffuson

In the UN limit, there exist the additional π -mode cooperon and diffuson due to the global particle-hole symmetry [23, 26] $\tau_2 G_{\mathbf{k}}^{R(A)}(\epsilon) \tau_2 = G_{\mathbf{Q}+\mathbf{k}}^{R(A)}(\epsilon)$, with $\mathbf{Q} = \pm(\pi/a, \pm\pi/a)$ the nesting vector. Any small deviations from this combined limit can be shown to make the Goldstone π -modes gapped. The ladder diagrams for the π -mode cooperon can be obtained from those of the 0-mode cooperon by replacing \mathbf{q} by $\mathbf{Q} + \mathbf{q}$. The equation for the π -mode cooperon is given by

$$\mathcal{A}_\pi(\mathbf{q}; \epsilon, \epsilon') \mathcal{C}_\pi(\mathbf{q}; \epsilon, \epsilon') = \mathcal{W}(\epsilon, \epsilon'), \quad (3.30)$$

where

$$\mathcal{A}_\pi(\mathbf{q}; \epsilon, \epsilon') = \mathcal{I} - \mathcal{W}(\epsilon, \epsilon') \mathcal{H}_\pi(\mathbf{q}; \epsilon, \epsilon'), \quad (3.31)$$

with

$$\mathcal{H}_\pi(\mathbf{q}; \epsilon, \epsilon')^{RR(A)} = \sum_{\mathbf{k}} G_{\mathbf{Q}+\mathbf{q}-\mathbf{k}}^R(\epsilon) \otimes G_{\mathbf{k}}^{R(A)}(\epsilon'). \quad (3.32)$$

In order to calculate $\mathcal{H}_\pi(\mathbf{q}; \epsilon, \epsilon')$ and $\mathcal{A}_\pi(\mathbf{q}; \epsilon, \epsilon')$, one needs to exploit the relations $\Delta_{\mathbf{Q}+\mathbf{k}} = -\Delta_{\mathbf{k}}$ and $\xi_{\mathbf{Q}+\mathbf{k}} = -\xi_{\mathbf{k}} - 2\mu$. For a nearly-nested Fermi surface ($|\mu| \ll \gamma$), we have

$$\begin{aligned} G_{\mathbf{Q}+\mathbf{q}-\mathbf{k}}^R(\epsilon) \approx & \tau_2 G_{\mathbf{q}-\mathbf{k}}^R(\epsilon) \tau_2 + 2\mu \frac{2\xi_{\mathbf{k}}(i\gamma\tau_0 - \Delta_{\mathbf{k}}\tau_1) + (\gamma^2 + \Delta_{\mathbf{k}}^2 - \xi_{\mathbf{k}}^2)\tau_3}{(\gamma^2 + \epsilon_{\mathbf{k}}^2)^2} \\ & + (2\mu)^2 \frac{(\gamma^2 + \Delta_{\mathbf{k}}^2 - 3\xi_{\mathbf{k}}^2)(i\gamma\tau_0 - \Delta_{\mathbf{k}}\tau_1) - \xi_{\mathbf{k}}(3\gamma^2 + 3\Delta_{\mathbf{k}}^2 - \xi_{\mathbf{k}}^2)\tau_3}{(\gamma^2 + \epsilon_{\mathbf{k}}^2)^3}. \end{aligned} \quad (3.33)$$

Substituting Eqs. (3.11) and (3.33) into Eq. (3.32), and using Eqs. (A1)–(A4), we can show that all the nonvanishing components of $\mathcal{H}_\pi(\mathbf{q}; \epsilon, \epsilon')$ are given by

$$H_\pi(\mathbf{q}; \epsilon, \epsilon')_{00}^{RR(A)} = H(\mathbf{q}; \epsilon, \epsilon')_{00}^{RR(A)} \pm \frac{\pi\rho_0\mu^2}{6l\gamma^3}, \quad (3.34)$$

$$H_\pi(\mathbf{q}; \epsilon, \epsilon')_{11}^{RR(A)} = -H(\mathbf{q}; \epsilon, \epsilon')_{11}^{RR(A)} + \frac{\pi\rho_0\mu^2}{12l\gamma^3}, \quad (3.35)$$

$$H_\pi(\mathbf{q}; \epsilon, \epsilon')_{33}^{RR(A)} = -H(\mathbf{q}; \epsilon, \epsilon')_{33}^{RR(A)} + \frac{\pi\rho_0\mu^2}{4l\gamma^3}, \quad (3.36)$$

and

$$H_\pi(\mathbf{q}; \epsilon, \epsilon')_{03}^{RR(A)} = \pm H_\pi(\mathbf{q}; \epsilon, \epsilon')_{30}^{RR(A)} = -i \frac{\pi\rho_0\mu}{4l\gamma^2}. \quad (3.37)$$

A substitution of Eqs. (3.6)–(3.9) and (3.34)–(3.37) into Eq. (3.31) yields the diagonal components of $\mathcal{A}_\pi(\mathbf{q}; \epsilon, \epsilon')$ as

$$\begin{aligned} A_\pi(\mathbf{q}; \epsilon, \epsilon')_{00}^{RR(A)} = & -\frac{1}{4l(1+\eta^2)} \left[-2(2l-1) - (6l-1)\eta^2 + i \frac{2\lambda - (1-\lambda)\eta^2}{\gamma} (\epsilon \pm \epsilon') \right. \\ & \left. - \frac{2v_f^2 + 2v_g^2 - (3v_f^2 + v_g^2)\eta^2}{12\gamma^2} q^2 - \frac{(4-6\eta^2)\mu^2}{3\gamma^2} - \frac{4\eta\mu}{\gamma} \right] \end{aligned} \quad (3.38)$$

$$A_\pi(\mathbf{q}; \epsilon, \epsilon')_{11}^{RR(A)} = \mp \frac{1}{4l(1+\eta^2)} \left[(2l-1) + i \frac{2\lambda(l-1)+1}{\gamma} (\epsilon \pm \epsilon') - \frac{v_f^2 + 3v_g^2}{12\gamma^2} q^2 - \frac{2\mu^2}{3\gamma^2} \right] \quad (3.39)$$

$$A_\pi(\mathbf{q}; \epsilon, \epsilon')_{22}^{RR(A)} = -\frac{\eta^2}{4l(1+\eta^2)} \left[(2l-1) + i\frac{1-\lambda}{\gamma}(\epsilon \pm \epsilon') - \frac{v_f^2 + 3v_g^2}{12\gamma^2}q^2 - \frac{2\mu^2}{3\gamma^2} \right], \quad (3.40)$$

and

$$A_\pi(\mathbf{q}; \epsilon, \epsilon')_{33}^{RR(A)} = \mp \frac{1}{4l(1+\eta^2)} \left[(2l-1) - 2\eta^2 + i\frac{2\lambda(l-1)+1}{\gamma}(\epsilon \pm \epsilon') - \frac{v_g^2 + 3v_f^2 - (2v_f^2 + 2v_g^2)\eta^2}{12\gamma^2}q^2 - \frac{(6-4\eta^2)\mu^2}{3\gamma^2} - \frac{4\eta\mu}{\gamma} \right]. \quad (3.41)$$

Like the 0-mode cooperon, the π -mode cooperon near the Born or unitary limit can be also expressed as $\mathcal{C}_\pi(\mathbf{q}; \epsilon, \epsilon') = \sum_i C_\pi(\mathbf{q}; \epsilon, \epsilon')_{ii} \tau_i \otimes \tau_i$, with all $C_\pi(\mathbf{q}; \epsilon, \epsilon')_{ii}$ assumed to be of diffusive poles. Therefore, Eqs. (3.19)–(3.22) are also suitable for the π -mode cooperon. Near the unitary limit ($\eta^2 \ll 1$), the singular terms in Eqs. (3.19)–(3.22) (for the π -mode cooperon) satisfy the following relations:

$$2C_{\pi 00}^{RR(A)} \mp C_{\pi 11}^{RR(A)} \mp C_{\pi 33}^{RR(A)} = 0, \quad (3.42)$$

$$2C_{\pi 11}^{RR(A)} \mp C_{\pi 00}^{RR(A)} \pm C_{\pi 22}^{RR(A)} = 0, \quad (3.43)$$

$$2C_{\pi 22}^{RR(A)} \pm C_{\pi 33}^{RR(A)} \pm C_{\pi 11}^{RR(A)} = 0, \quad (3.44)$$

$$2C_{\pi 33}^{RR(A)} \pm C_{\pi 22}^{RR(A)} \mp C_{\pi 00}^{RR(A)} = 0, \quad (3.45)$$

the solution of which is shown to be

$$C_{\pi 00}^{RR(A)} = \pm C_{\pi 11}^{RR(A)} = -C_{\pi 22}^{RR(A)} = \pm C_{\pi 33}^{RR(A)}. \quad (3.46)$$

Substituting Eqs. (3.6), (3.38)–(3.41) and (3.46) into Eq. (3.19), and using Eq. (2.6), we can show that

$$C_\pi(\mathbf{q}; \epsilon, \epsilon')_{00}^{RR(A)} = \mp \frac{4\gamma^2}{\pi\rho_0} \frac{1}{Dq^2 - i(\epsilon \pm \epsilon') + 2\delta}, \quad (3.47)$$

with $\delta = 2\eta^2\gamma + 2\eta\mu/\gamma + \mu^2/l\gamma \ll \gamma$. The present expression of δ supports the result estimated by Yashenkin et al. [23]. Combining Eq. (3.46) with Eq. (3.47), and noting

that the π -mode diffuson has also the same expression as that of the π -mode cooperon due to the time-reversal symmetry, we obtain

$$\begin{aligned} \mathcal{C}_\pi(\mathbf{q}; \epsilon, \epsilon')^{RR(A)} &= \mathcal{D}_\pi(\mathbf{q}; \epsilon, \epsilon')^{RR(A)} \\ &= -\frac{4\gamma^2}{\pi\rho_0} \frac{1}{Dq^2 - i(\epsilon \pm \epsilon') + 2\delta} \left(\pm \tau_0 \otimes \tau_0 + \tau_1 \otimes \tau_1 \mp \tau_2 \otimes \tau_2 + \tau_3 \otimes \tau_3 \right). \end{aligned} \quad (3.48)$$

Near the Born limit ($\eta^2 \gg 2l$), one can easily show that $C_{\pi ii}^{RR(A)} = 0$, ($i = 0, 1, 2, 3$), indicating that the diffusive π -modes exist only near the UN limit. Contrary to the diffusive 0-modes, the Goldstone π -modes are gapped by any small deviations from the UN limit measured by δ . For the situations far from the UN limit, the contributions of diffusive π -modes to the QI effect are completely suppressed due to the large gap.

IV. QI correction to the quasiparticle DOS

The QI correction to the quasiparticle DOS can be calculated via

$$\Delta\rho(\epsilon) = -\frac{1}{\pi} \text{Im} \sum_{\mathbf{k}} \text{Tr} \Delta G_{\mathbf{k}}^R(\epsilon), \quad (4.1)$$

where $\Delta G_{\mathbf{k}}^R(\epsilon)$ represents the lowest-order correction to the one-particle Green's function due to the diffusive modes.

A. Contribution of the 0-mode cooperon

The lowest-order self-energy diagrams with 0-mode cooperon are depicted in Fig. 1. The contribution of Fig. 1(a) has been calculated in Ref. [23]. Figure 1(b) contains the non-singular ladders, its contribution to the DOS can be shown to be the same order of magnitude as that of Fig. 1(a). According to Eq. (4.1), the contribution of Fig. 1(a) to the DOS reads

$$\rho(\epsilon)_a = -\frac{1}{\pi} \text{Im} \sum_{\mathbf{k}\mathbf{q}} \sum_i C(\mathbf{q}; \epsilon, \epsilon)_{ii}^{RR} \text{Tr} (\tau_i G_{\mathbf{q}-\mathbf{k}}^R \tau_i G_{\mathbf{k}}^R G_{\mathbf{k}}^R).$$

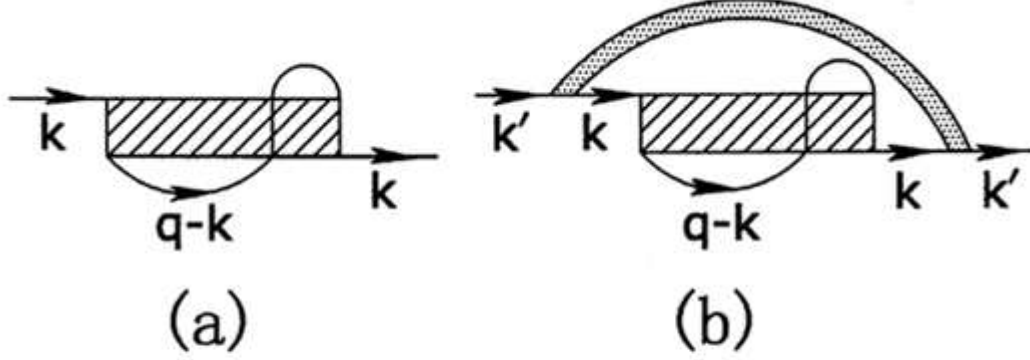


Figure 1: The lowest-order self-energy diagrams with 0-mode cooperon (shaded blocks). The grey block in Fig. 1(b) represents the non-singular ladders. The corresponding diagrams with π -mode cooperon can be obtained by the replacement of $\mathbf{q} \rightarrow \mathbf{Q} + \mathbf{q}$ in Figs. 1(a) and 1(b).

By means of Eq. (A5), it can be rewritten as

$$\rho(\epsilon)_a = -\frac{2}{\pi} \text{Im} \sum_{\mathbf{q}} \sum_i [\mathcal{C}(\mathbf{q}; \epsilon, \epsilon)^{RR} \mathcal{M}_a]_{ii},$$

with $\mathcal{M}_a = \sum_{\mathbf{k}} G_{\mathbf{k}}^R \otimes (G_{\mathbf{k}}^R G_{\mathbf{k}}^R)$. The contribution of Fig. 1(b) can be similarly evaluated, and the total contribution of Figs. 1(a) and 1(b) can be expressed by

$$\rho(\epsilon)_{\text{coop}} = -\frac{2}{\pi} \text{Im} \sum_{\mathbf{q}} \sum_i [\mathcal{C}(\mathbf{q}; \epsilon, \epsilon)^{RR} \mathcal{M}]_{ii}, \quad (4.2)$$

with

$$\mathcal{M} = \sum_{\mathbf{k}} G_{\mathbf{k}}^R \otimes (G_{\mathbf{k}}^R K G_{\mathbf{k}}^R). \quad (4.3)$$

Here the matrix K satisfies

$$K = \tau_0 + n_i \sum_{\mathbf{k}} T^R G_{\mathbf{k}}^R K G_{\mathbf{k}}^R T^R, \quad (4.4)$$

with $T^{R(A)} = T^{R(A)}(0)$. Substituting Eqs. (2.1) and (2.2) into Eq. (4.4), and using Eqs. (A1)–(A4), we can show that

$$K = (l/\zeta)\tau_0, \quad (4.5)$$

where

$$\zeta = \begin{cases} 1, & \text{for the Born limit,} \\ 2l - 1, & \text{for the unitary limit.} \end{cases} \quad (4.6)$$

Equation (4.5) leads to $\mathcal{M} = (l/\zeta)\mathcal{M}_a$, and hence $\rho(\epsilon)_{\text{coop}} = (l/\zeta)\rho(\epsilon)_a$, indicating that the contribution of Fig. 1(b) just renormalizes the pre-factor of $\rho(\epsilon)_a$. A substitution of Eqs. (2.2) and (4.5) into Eq. (4.3) yields

$$\mathcal{M} = -i \frac{l}{2\pi\zeta v_f v_g \gamma} (\tau_1 \otimes \tau_1 + \tau_3 \otimes \tau_3). \quad (4.7)$$

Substituting Eqs. (3.29) and (4.7) into Eq. (4.2), and noting that the upper cutoff of q can be set to be $1/l_e$ with $l_e = \sqrt{D/2\gamma}$ the mean-free path, we obtain

$$\begin{aligned} \frac{\rho(\epsilon)_{\text{coop}}}{\rho_0} &= -\frac{2\pi}{\alpha\zeta} \text{Re} \sum_{\mathbf{q}} \frac{D}{Dq^2 - i2\epsilon} \\ &= -\frac{1}{2\alpha\zeta} \ln \frac{\gamma}{|\epsilon|}. \end{aligned} \quad (4.8)$$

where $\alpha = (v_f^2 + v_g^2)/2v_f v_g$.

B. Vanishing contribution of the 0-mode diffuson

The lowest-order self-energy diagrams with the 0-mode diffuson are given by Figs. 2(a) and 2(b), and the vertex correction to the impurity-scattering T -matrix is shown by Fig. 2(c). Unlike the single-vertex-dressed diagram used in Ref. [23], the present Figs. 2(a) and 2(b) include the additional diagrams with *both* vertices dressed by the π -mode diffuson. Similar vertex correction has been considered in the theory of disordered interacting-electron systems [29]. The vertex-corrected retarded T -matrix can be expressed by

$$\bar{T}^R(\mathbf{q}, \epsilon)_{\mu\mu'} = \sum_{\nu\nu'} \mathcal{J}(\mathbf{q}, \epsilon)_{\mu\mu', \nu\nu'}^{RR} T^R(\epsilon)_{\nu\nu'}, \quad (4.9)$$

where μ, μ', ν , and ν' are the indices of the Nambu space, and the vertex function $\mathcal{J}(\mathbf{q}, \epsilon)^{RR}$ is given by

$$\mathcal{J}(\mathbf{q}, \epsilon)^{RR} = \mathcal{I} + \mathcal{D}(\mathbf{q}; \epsilon, \epsilon)^{RR} \mathcal{H}(\mathbf{q}; \epsilon, \epsilon)^{RR}. \quad (4.10)$$

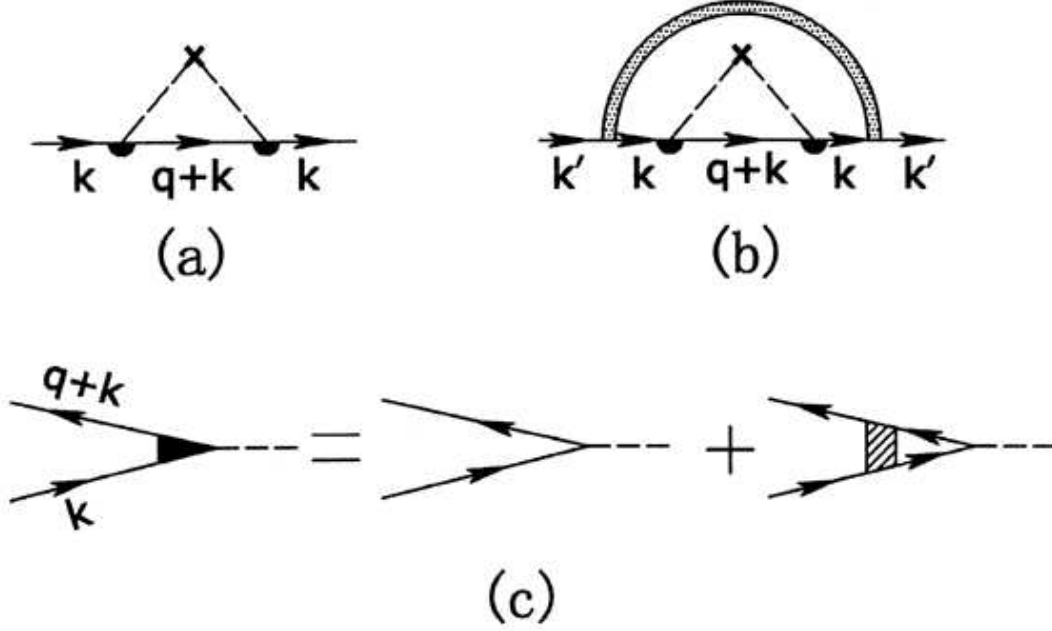


Figure 2: The lowest-order self-energy diagrams with 0-mode diffuson (a and b), and the vertex diagram (c). The shaded and grey blocks represent, respectively, the 0-mode diffuson and non-singular ladders. The dashed lines denote the impurity-scattering T -matrix. The corresponding diagrams with π -mode diffuson can be generated by the replacement of $\mathbf{q} \rightarrow \mathbf{Q} + \mathbf{q}$ in these 0-mode diagrams.

In order to calculate the vertex function, we exploit the equation for 0-mode diffuson in the RR-channel

$$\mathcal{D}(\mathbf{q}; \epsilon, \epsilon)^{RR} = \mathcal{W}(\epsilon, \epsilon)^{RR} + \mathcal{W}(\epsilon, \epsilon)^{RR} \mathcal{H}(\mathbf{q}; \epsilon, \epsilon)^{RR} \mathcal{D}(\mathbf{q}; \epsilon, \epsilon)^{RR}, \quad (4.11)$$

yielding

$$\mathcal{H}(\mathbf{q}; \epsilon, \epsilon)^{RR} = \mathcal{W}^{-1}(\epsilon, \epsilon)^{RR} - \mathcal{D}^{-1}(\mathbf{q}; \epsilon, \epsilon)^{RR}. \quad (4.12)$$

A substitution of Eq. (4.12) into Eq. (4.10) leads to

$$\mathcal{J}(\mathbf{q}, \epsilon)^{RR} = \mathcal{D}(\mathbf{q}; \epsilon, \epsilon)^{RR} \mathcal{W}^{-1}(\epsilon, \epsilon)^{RR}. \quad (4.13)$$

Making use of Eqs. (3.6) and (3.7), we get

$$\mathcal{W}^{-1}(\epsilon, \epsilon)^{RR} \approx \frac{\pi \rho_0}{2\gamma} \times \begin{cases} \tau_3 \otimes \tau_3, & \text{for the Born limit,} \\ -\tau_0 \otimes \tau_0, & \text{for the unitary limit.} \end{cases} \quad (4.14)$$

Substituting Eqs. (3.29) and (4.14) into Eq. (4.13), we obtain the expression of the vertex function as (for $Dq^2 \ll \gamma$ and $|\epsilon| \ll \gamma$)

$$\mathcal{J}(\mathbf{q}, \epsilon)^{RR} = \frac{2\gamma}{Dq^2 - i2\epsilon} (\tau_0 \otimes \tau_0 - \tau_1 \otimes \tau_1 - \tau_2 \otimes \tau_2 - \tau_3 \otimes \tau_3), \quad (4.15)$$

which is suitable both near the Born and near the unitary limits. Substituting Eqs. (2.1) and (4.15) into Eq. (4.9), we can easily show that

$$\bar{T}^R(\mathbf{q}, \epsilon) = \sum_i J(\mathbf{q}, \epsilon)_{ii}^{RR} \tau_i T^R(\epsilon) \tau_i^* = 0, \quad (4.16)$$

indicating that both Figs. 2(a) and 2(b) have vanishing contributions to the DOS near the Born or unitary limit.

C. Contribution of the π -mode cooperon near the UN limit

Near the UN limit, besides the diffusive 0-modes, the Goldstone π -modes also contribute to the QI effect. The leading self-energy diagrams with the π -mode cooperon can be obtained from those in Fig. 1 by a replacement of $\mathbf{q} \rightarrow \mathbf{q} + \mathbf{Q}$. Similarly with the case of 0-mode cooperon, the contribution of π -mode cooperon to the DOS can be evaluated via

$$\rho(\epsilon)_{\pi\text{-coop}} = -\frac{2}{\pi} \text{Im} \sum_{\mathbf{q}} \sum_i [\mathcal{C}_{\pi}(\mathbf{q}; \epsilon, \epsilon)^{RR} \mathcal{M}_{\pi}]_{ii}, \quad (4.17)$$

where

$$\mathcal{M}_{\pi} = \sum_{\mathbf{k}} G_{\mathbf{Q}-\mathbf{k}}^R \otimes (G_{\mathbf{k}}^R K G_{\mathbf{k}}^R), \quad (4.18)$$

with $K = l\tau_0/(2l-1)$ for the unitary limit. Expression (4.18) can be readily evaluated by means of the global particle-hole symmetry, yielding

$$\mathcal{M}_{\pi} = i \frac{l}{2\pi(2l-1)v_f v_g \gamma} (\tau_1 \otimes \tau_1 + \tau_3 \otimes \tau_3). \quad (4.19)$$

Substituting Eqs. (3.48) and (4.19) into Eq. (4.17), we obtain

$$\begin{aligned} \frac{\rho(\epsilon)_{\pi\text{-coop}}}{\rho_0} &= \frac{2\pi}{\alpha(2l-1)} \text{Re} \sum_{\mathbf{q}} \frac{D}{Dq^2 - i2\epsilon + 2\delta} \\ &= \frac{1}{2\alpha(2l-1)} \ln \frac{\gamma}{\sqrt{\epsilon^2 + \delta^2}}. \end{aligned} \quad (4.20)$$

Comparing Eq. (4.8) with Eq. (4.20), one finds that the contribution to DOS of the π -mode cooperon has an equal magnitude but an opposite sign to that of the 0-mode cooperon in the UN limit ($\delta \rightarrow 0$).

D. Contribution of the π -mode diffuson near the UN limit

The leading self-energy diagrams containing the π -mode diffuson are generated by replacing \mathbf{q} by $\mathbf{Q} + \mathbf{q}$ in Figs. 2(a) and 2(b). Obviously, Eqs. (4.9) and (4.13) are also suitable for the π -mode diffuson. By exploiting Eq. (3.48), we can readily show that

$$\mathcal{J}_\pi(\mathbf{q}, \epsilon)^{RR} = \frac{2\gamma}{Dq^2 - i2\epsilon + 2\delta} \left(\tau_0 \otimes \tau_0 + \tau_1 \otimes \tau_1 - \tau_2 \otimes \tau_2 + \tau_3 \otimes \tau_3 \right) \quad (4.21)$$

and

$$\begin{aligned} \bar{T}_\pi^R(\mathbf{q}, \epsilon) &= \sum_i J_\pi(\mathbf{q}, \epsilon)_{ii}^{RR} \tau_i T^R(\epsilon) \tau_i^* \\ &= -i \frac{16\gamma}{\pi \rho_0} \frac{1}{Dq^2 - i2\epsilon + 2\delta} \tau_0, \end{aligned} \quad (4.22)$$

which are valid near the UN limit.

The contribution of the π -mode diffuson to the DOS is given by

$$\rho(\epsilon)_{\pi\text{-diff}} = -\frac{n_i}{\pi} \text{Im} \sum_{\mathbf{k}\mathbf{q}} \text{Tr} \left[\bar{T}_\pi^R(\mathbf{q}, \epsilon) G_{\mathbf{Q}+\mathbf{k}}^R \bar{T}_\pi^R(\mathbf{q}, \epsilon) G_{\mathbf{k}}^R K G_{\mathbf{k}}^R \right]. \quad (4.23)$$

Substituting Eqs. (2.2) and (4.22) into Eq. (4.23), and using Eqs. (2.6) and (A1)–(A4), one can easily show that

$$\begin{aligned} \frac{\rho(\epsilon)_{\pi\text{-diff}}}{\rho_0} &= \frac{32\pi\gamma}{\alpha(2l-1)} \text{Re} \sum_{\mathbf{q}} \frac{D}{(Dq^2 - i2\epsilon + 2\delta)^2} \\ &= \frac{4}{\alpha(2l-1)} \frac{\gamma\delta}{\epsilon^2 + \delta^2}. \end{aligned} \quad (4.24)$$

Expression (4.24) is quite different from the logarithmic behavior obtained in Ref. [23], due to the additional contributions of the self-energy diagrams with both the impurity-scattering vertices dressed by the π -mode cooperon.

E. Results of the QI correction to DOS

For generic Fermi surfaces, the QI correction to the DOS results only from the contribution of 0-mode cooperon, i.e.,

$$\frac{\Delta\rho(\epsilon)}{\rho_0} = \frac{\rho(\epsilon)_{\text{coop}}}{\rho_0} = -\frac{1}{2\alpha\zeta} \ln \frac{\gamma}{|\epsilon|}, \quad (4.25)$$

where ζ is given by Eq. (4.6). Therefore, the QI effect gives rise to a logarithmic suppression to the quasiparticle DOS, as predicted by Yashenkin et al. [23]. Equation (4.25) is suitable near the Born or unitary limit. In the intermediate region far from these two limits, however, the 0-mode cooperon may contain some singular non-diagonal components, and thus a refined theory is necessary for the generic situations. In addition, the additional contributions of the self-energy diagrams with non-singular ladders yield various renormalization factors in these two limits.

Near the UN limit, the QI correction to the DOS is the sum of $\rho(\epsilon)_{\text{coop}}$, $\rho(\epsilon)_{\pi\text{-coop}}$, and $\rho(\epsilon)_{\pi\text{-diff}}$, so that

$$\frac{\Delta\rho(\epsilon)_{\text{UN}}}{\rho_0} = \frac{1}{2\alpha(2l-1)} \left[-\ln \frac{\gamma}{|\epsilon|} + \ln \frac{\gamma}{\sqrt{\epsilon^2 + \delta^2}} + \frac{8\gamma\delta}{\epsilon^2 + \delta^2} \right]. \quad (4.26)$$

The above equation indicates that the quasiparticle DOS near the UN limit can be subject to a positive correction due to the contributions of the Goldstone π -modes. Upon approaching the UN limit (δ is small enough), the contributions from the 0-mode and π -mode cooperons cancel out each other, so that the DOS correction is given only by the contribution of the π -mode diffuson, i.e.,

$$\lim_{\delta \rightarrow 0} \frac{\Delta\rho(\epsilon)_{\text{UN}}}{\rho_0} = \frac{4\pi\gamma}{\alpha(2l-1)} \delta(\epsilon). \quad (4.27)$$

This result can be used to account for the sharp peak found in the numerical studies of the DOS [19, 20].

V. Summary

We have extensively investigated the QI effect on the quasiparticle DOS in a $d_{x^2-y^2}$ -wave superconductor with dilute nonmagnetic impurities. As in the study of disordered normal metals [24], the understanding of the diffusive modes is essential for the investigation of the QI effect in a superconductor. Through detailed derivations, we have obtained the expressions for the diffusive modes both near the Born and near the unitary limits. It is demonstrated that these Goldstone modes may contain non-diagonal components in the intermediate region far from these two limits. Therefore, a refined weak-localization theory is necessary for the generic situations.

For generic Fermi surfaces, the QI effect results only from the 0-mode cooperon, yielding a logarithmic suppression to the quasiparticle DOS near the Born or unitary limit. We show that those non-trivial self-energy diagrams containing non-singular ladders give rise to various renormalization factors of the DOS correction for the Born and unitary limits.

Near the UN (unitary and nesting) limit, the QI effect comes not only from the contribution of the 0-mode cooperon, but also from those of the π -mode cooperon and diffuson. It is found that the self-energy diagrams with both impurity-scattering vertices corrected by the π -mode diffuson have an important contribution to the DOS, which is proportional to $\delta/(\epsilon^2 + \delta^2)$. As a result, the quasiparticle DOS is subject to a positive correction induced by the diffusive π -modes. Upon approaching the UN limit ($\delta \rightarrow 0$), the contributions of the 0-mode and π -mode cooperons cancel out each other (even including the renormalization contributions induced by the non-singular ladders), so that the DOS correction becomes a δ -function of the energy. This result can be used to account for the resonant peak found in the previous numerical studies [19, 20]. A similar sharp peak of electronic DOS has been found numerically in a disordered 2D tight-binding model for the normal state [30]. The appearance of this sharp peak can be also explained by the contribution of Fig. 2(a) with the π -mode diffuson (Fig. 2(b) has a vanishing contribution for the normal state) [31].

Like in the case of disordered interacting-electron systems [29], all the leading polarization diagrams responsible for the QI effect on the quasiparticle conductivity can be

generated in the conserving approximation from the lowest-order self-energy diagrams shown in Figs. 1 and 2. For the normal state, it has been shown that the contributions of those polarization diagrams with π -mode diffuson lead to an antilocalization correction to the conductivity in the UN limit [32]. How the QI processes related with the diffusive π -modes affect the transport properties, such as electrical and spin conductivities, in a d -wave superconductor is another interesting problem, and will be studied in our future work.

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Appendix: Some useful mathematical formulas

In this appendix, we present some mathematical formulas, which are useful for the evaluations in the previous sections. By the approach used in Ref. [33], we can show that

$$\sum_{\mathbf{k}} \frac{\xi_{\mathbf{k}}^2 \Delta_{\mathbf{k}}^2}{(\gamma^2 + \epsilon_{\mathbf{k}}^2)^4} = \frac{1}{24\pi v_f v_g \gamma^2}, \quad (\text{A1})$$

$$\sum_{\mathbf{k}} \frac{\xi_{\mathbf{k}}^4}{(\gamma^2 + \epsilon_{\mathbf{k}}^2)^4} = \sum_{\mathbf{k}} \frac{\Delta_{\mathbf{k}}^4}{(\gamma^2 + \epsilon_{\mathbf{k}}^2)^4} = \frac{1}{8\pi v_f v_g \gamma^2}, \quad (\text{A2})$$

$$\sum_{\mathbf{k}} \frac{1}{(\gamma^2 + \epsilon_{\mathbf{k}}^2)^n} = \frac{1}{(n-1)\pi v_f v_g \gamma^{2(n-1)}}, \quad \text{for } n \geq 2, \quad (\text{A3})$$

and

$$\sum_{\mathbf{k}} \frac{\xi_{\mathbf{k}}^2}{(\gamma^2 + \epsilon_{\mathbf{k}}^2)^n} = \sum_{\mathbf{k}} \frac{\Delta_{\mathbf{k}}^2}{(\gamma^2 + \epsilon_{\mathbf{k}}^2)^n} = \frac{1}{2(n-1)(n-2)\pi v_f v_g \gamma^{2(n-2)}}, \quad \text{for } n \geq 3. \quad (\text{A4})$$

As an example, we shall prove Eq. (A1). Using the Dirac-type quasiparticle spectrum, and noting that there exist four gap nodes, we have

$$\sum_{\mathbf{k}} \frac{\xi_{\mathbf{k}}^2 \Delta_{\mathbf{k}}^2}{(\gamma^2 + \epsilon_{\mathbf{k}}^2)^4} = 4 \int \int \frac{d\tilde{k}_f d\tilde{k}_g}{(2\pi)^2} \frac{v_f^2 v_g^2 \tilde{k}_f^2 \tilde{k}_g^2}{(\gamma^2 + v_f^2 \tilde{k}_f^2 + v_g^2 \tilde{k}_g^2)^4}.$$

By means of the transformations of $p_f = \sqrt{v_f/v_g} \tilde{k}_f$ and $p_g = \sqrt{v_g/v_f} \tilde{k}_g$, the above equation can be changed as

$$\begin{aligned} \sum_{\mathbf{k}} \frac{\xi_{\mathbf{k}}^2 \Delta_{\mathbf{k}}^2}{(\gamma^2 + \epsilon_{\mathbf{k}}^2)^4} &= 4 \int \int \frac{dp_f dp_g}{(2\pi)^2} \frac{v_f^2 v_g^2 p_f^2 p_g^2}{[\gamma^2 + v_f v_g (p_f^2 + p_g^2)]^4} \\ &= \frac{1}{2\pi^2 v_f v_g \gamma^2} \int_0^{x_0} dx \frac{x^2}{(1+x)^4} \int_0^{2\pi} d\theta \cos^2 \theta \sin^2 \theta, \end{aligned}$$

where $x_0 = v_f v_g p_0^2 / \gamma^2$ with $p_0 \sim 1/a$. For the weak-disorder case (γ is small enough) under considered, we can set $x_0 = \infty$. Thus, a completion of the integrals over x and θ in the above equation immediately yields Eq. (A1).

Another useful formula is given by

$$\frac{1}{2} \sum_i C_{ii} \text{Tr}(\tau_i A \tau_i B) = \sum_i (\mathcal{CM})_{ii}, \quad (\text{A5})$$

where $\mathcal{C} = \sum_i C_{ii} \tau_i \otimes \tau_i$, and $\mathcal{M} = A \otimes B$ with A and B being any linear superimpositions of τ_i ($i=0,1,2,3$). In order to prove Eq. (A5), we assum that $\tau_i A = \sum_j x_{ij} \tau_j$ and $\tau_i B = \sum_k y_{ik} \tau_k$. Then we have

$$\frac{1}{2} \sum_i C_{ii} \text{Tr}(\tau_i A \tau_i B) = \frac{1}{2} \sum_{ijk} C_{ii} x_{ij} y_{ik} \text{Tr}(\tau_j \tau_k) = \sum_{ijk} C_{ii} x_{ij} y_{ik} \delta_{jk} = \sum_{ij} C_{ii} x_{ij} y_{ij} \quad (\text{A6})$$

and

$$\mathcal{CM} = \sum_i C_{ii} (\tau_i A) \otimes (\tau_i B) = \sum_{ijk} C_{ii} x_{ij} y_{ik} \tau_j \otimes \tau_k. \quad (\text{A7})$$

From Eq. (A7), it follows that

$$\sum_i (\mathcal{CM})_{ii} = \sum_{ij} C_{ii} x_{ij} y_{ij}. \quad (\text{A8})$$

A combination of Eq. (A6) with Eq. (A8) immediately leads to Eq. (A5).

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